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ON THE PERIODIC CAUCHY PROBLEM FOR A COUPLED SYSTEM OF THIRD-ORDER NONLINEAR SCHRÖDINGER EQUATIONS

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ABSTRACT. We investigate some well-posedness issues for the initial value problem (IVP) associated to the system

$$\begin{cases} 2i\partial_t u + q\partial_x^2 u + i\gamma\partial_x^3 u = F_1(u, w) \\ 2i\partial_t w + q\partial_x^2 w + i\gamma\partial_x^3 w = F_2(u, w), \end{cases}$$

where F_1 and F_2 are polynomials of degree 3 involving u , w and their derivatives. This system describes the dynamics of two nonlinear short-optical pulses envelopes $u(x, t)$ and $w(x, t)$ in fibers ([13], [8]). We prove periodic local well-posedness for the IVP with data in Sobolev spaces $H^s(\mathbb{T}) \times H^s(\mathbb{T})$, $s \geq 1/2$ and global well-posedness result in Sobolev spaces $H^1(\mathbb{T}) \times H^1(\mathbb{T})$.

1. INTRODUCTION

Consider the initial value problem (IVP)

$$\begin{cases} 2i\partial_t u + q\partial_x^2 u + i\gamma\partial_x^3 u = F_1(u, w), & x \in \mathbb{T}, t > 0, \\ 2i\partial_t w + q\partial_x^2 w + i\gamma\partial_x^3 w = F_2(u, w), \\ u(x, 0) = u_0, & w(x, 0) = w_0, \end{cases} \quad (1.1)$$

where u , w are complex valued functions,

$$F_1(u, w) = -2i\beta(|u|^2 + \sigma_\beta|w|^2)\partial_x u - 2\alpha u(|u|^2 + \sigma_\alpha|w|^2) - 2i\mu u\partial_x(|u|^2 + \sigma_\mu|w|^2),$$

$$F_2(u, w) = F_1(w, u) \text{ and } q, \gamma, \beta, \mu, \alpha, \sigma_\alpha, \sigma_\beta \text{ and } \sigma_\mu \text{ are real parameters.}$$

This system describes the dynamics of two nonlinear short-optical pulses envelopes $u(x, t)$ and $w(x, t)$ in fibers. This model is formed by a pair of Schrödinger-Korteweg-de Vries equations coupled through nonlinear terms and it was derived by Porsezian, Shanmugha Sundaram and Mahalingam [13]. It generalizes the model (1.3) derived by Hasegawa-Kodama [8].

There is a growing interest in studying the propagation of optical soliton pulses in fiber. This is because of their potential applications in fiber-optic-based communication systems. The idea of soliton based all-optical communication systems with loss compensated by optical amplifications has provided hints of potential advantage for solitons over conventional systems. The major attraction for the soliton communication system arises from the fact that repeater spacing for this kind of system could be much larger than that required by the conventional systems.

The system above has been previously studied ([12] and [18]) in the particular case $\sigma_\alpha = \sigma_\beta = \sigma_\mu = 1$ and the system of Hirota and Hirota-Satsuma studied by

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[2] and [1] respectively. Radhakrishnan and Lakshmanan [15] used the following transformation of variables in system (1.1)

$$\begin{aligned} u(x, t) &= u_1 \left(x - \frac{q^2}{6\gamma}t, t \right) \exp i \left(\frac{q}{3\gamma}x - \frac{q^3}{27\gamma^2}t \right), \\ w(x, t) &= w_1 \left(x - \frac{q^2}{6\gamma}t, t \right) \exp i \left(\frac{q}{3\gamma}x - \frac{q^3}{27\gamma^2}t \right), \end{aligned}$$

under the particular conditions $\sigma_\alpha = \sigma_\beta$ and $q\beta = 3\gamma\alpha$, to obtain the following form of coupled envelope equations corresponding to the system (1.1)

$$\begin{cases} 2\partial_t u_1 + \gamma \partial_x^3 u_1 + 2\beta (|u_1|^2 + \sigma_\beta |w_1|^2) \partial_x u_1 + 2\mu u_1 \partial_x (|u_1|^2 + \sigma_\mu |w_1|^2) = 0, \\ 2\partial_t w_1 + \gamma \partial_x^3 w_1 + 2\beta (|w_1|^2 + \sigma_\beta |u_1|^2) \partial_x w_1 + 2\mu w_1 \partial_x (|w_1|^2 + \sigma_\mu |u_1|^2) = 0. \end{cases} \quad (1.2)$$

Then, they applied the Hirota bilinear transformation (see [9]) to (1.2) to construct dark and bright soliton solutions to (1.1) assuming further that $\beta = \mu$, $q\beta = 3\gamma\alpha$, $\gamma \neq 0$ and $\sigma_\alpha = \sigma_\beta = \sigma_\mu = 1$. Recently, Porsezian and Kalithasan [14] discussed the construction of new cnoidal wave solutions and found exact solutions of both bright and dark solitary wave to system (1.1).

As far as we know, the previous works in this subject do not address well-posedness issues for the system (1.1), so our aim is to fill up this gap.

Note that if the pulse $w_1 = 0$, the system (1.2) reduces to the well known modified complex KdV equation. This fact suggests that the results obtained for the periodic modified KdV equation should be the ones we expect for the system (1.1).

When $w = 0$ the system (1.1) reduces to equation

$$i\partial_t u + \frac{q}{2}\partial_x^2 u + i\frac{\gamma}{2}\partial_x^3 u + \alpha u|u|^2 + i(\beta + \mu)|u|^2\partial_x u + i\mu u^2\partial_x \bar{u} = 0, \quad (1.3)$$

which describes the dynamics of one single nonlinear pulse in an optic fiber.

The initial value problem associated to equation (1.3) was considered by several authors ([10], [16], [5], [17], [6] and [7]) in $H^s(\mathbb{R})$, where $s \geq 1/4$ is the best result.

In the case of system (1.1), Scialom and Bragança [4] obtained local well-posedness solution in Sobolev spaces $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s \geq 1/4$, and global well-posedness in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > 3/5$.

In the periodic setting, Takaoka in [17], considered the IVP (1.3) and showed local well-posedness in $H^s(\mathbb{T})$, $s \geq \frac{1}{2}$.

For the IVP associated to system (1.1), we are able to obtain local well-posedness for initial data in $H^s(\mathbb{T}) \times H^s(\mathbb{T})$, $s \geq \frac{1}{2}$ as in the single equation case. The approach we use is similar to the one given in [17] though the presence of the coupled terms in (1.1) make the estimates more involved.

To describe our local result we need the definitions and notations.

Definition 1.1. Let $\mathcal{P} = \mathcal{C}^\infty(\mathbb{T}) = \{g : \mathbb{R} \rightarrow \mathbb{C}; g \in \mathcal{C}^\infty \text{ periodic with period } 2\pi\}$. \mathcal{P}' (dual of \mathcal{P}) is the collection of all linear functionals from \mathcal{P} to \mathbb{C} . \mathcal{P}' is periodic distributions. If $g \in \mathcal{P}'$ denote the value of g in φ by $g(\varphi) = \langle g, \varphi \rangle$. Consider the functions $\theta_n(x) = e^{inx}$, $n \in \mathbb{Z}$ and $x \in \mathbb{R}$. The Fourier transform $g \in \mathcal{P}'$ is the function $\hat{g} : \mathbb{Z} \rightarrow \mathbb{C}$ defined by $\hat{g}(n) = \langle g, \theta_{-n} \rangle$. If g is periodic of period 2π , for instance, $g \in L^1(\mathbb{T})$ then

$$\hat{g}(n) = \int_{\mathbb{T}} e^{-inx} g(x) dx = \frac{1}{2\pi} \int_0^{2\pi} e^{-inx} g(x) dx,$$

where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ represents the one dimensional torus.

Definition 1.2. Let $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{T})$ is the set of all $g \in \mathcal{P}'$ such that

$$\|g\|_{H^s(\mathbb{T})} = \left(2\pi \sum_{n \in \mathbb{Z}} (1 + |n|^2)^s |\widehat{g}(n)|^2 \right)^{\frac{1}{2}} < \infty.$$

In this work we assume that g is periodic of period 2π .

Definition 1.3. We denote by \widetilde{f} the Fourier transform of f in relation to space-time variables

$$\widetilde{f}(n, \tau) = \int_{\mathbb{R}} \int_{\mathbb{T}} e^{-i(xn+t\tau)} f(x, t) dx dt.$$

The inverse Fourier transform is given by

$$f(x, t) = \sum_{n \in \mathbb{Z}} e^{inx} \int_{\mathbb{R}} e^{it\tau} \widetilde{f}(n, \tau) d\tau.$$

The Fourier transform of fgh , where $f = f(x, t)$, $g = g(x, t)$ and $h = h(x, t)$ are periodic functions with respect to the x variable obtained as

$$\widetilde{fgh}(n, \tau) = (\widetilde{f} * \widetilde{g} * \widetilde{h})(n, \tau) = \sum_{n_1 \in \mathbb{Z}} \sum_{n_2 \in \mathbb{Z}} \iint_{\mathbb{R}^2} \widetilde{f}(n_1, \tau_1) \widetilde{g}(n_2, \tau_2) \widetilde{h}(n - n_1 - n_2, \tau - \tau_1 - \tau_2) d\tau_1 d\tau_2.$$

Definition 1.4. Let \mathcal{V} be the space of functions f such that

- (i) $f : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{C}$,
- (ii) $f(x, \cdot) \in \mathcal{S}(\mathbb{R})$ for each $x \in \mathbb{T}$,
- (iii) $f(\cdot, t) \in C^\infty(\mathbb{T})$ for each $t \in \mathbb{R}$.

We define the space $X_{s,b}$ associated to operator $\partial_t - i\frac{q}{2}\partial_x^2 + \partial_x^3 + c_0\partial_x$ as the completion of \mathcal{V} with respect to the following norm

$$\|f\|_{X_{s,b}} := \|f\|_{(1,s,b)} = \|\langle n \rangle^s \langle \tau - n^3 + \frac{q}{2}n^2 + c_0n \rangle^b \widetilde{f}(n, \tau)\|_{l_n^2 L_\tau^2},$$

where $\langle n \rangle = (1 + |n|^2)^{\frac{1}{2}}$ and $s, b, c_0 \in \mathbb{R}$.

The space $Z_{s,b}$ is the completion of \mathcal{V} with respect to the norm

$$\|f\|_{Z_{s,b}} := \|f\|_{(2,s,b)} = \|\langle n \rangle^s \langle \tau - n^3 + \frac{q}{2}n^2 + c_0n \rangle^b \widetilde{f}(n, \tau)\|_{l_n^2 L_\tau^1},$$

and we consider the space $Y_s = X_{s, \frac{1}{2}} \cap Z_{s,0}$ with the norm

$$\|f\|_{Y_s} = \|f\|_{(1,s,\frac{1}{2})} + \|f\|_{(2,s,0)}.$$

Remark 1.5. Note that for $b > \frac{1}{2}$ we have that $X_{s,b} \subset C(\mathbb{R}_t; H^s(\mathbb{T}))$ and for $b = 0$, we have $Z_{s,0} \subset C(\mathbb{R}_t; H^s(\mathbb{T}))$ and $Y_s \subset C(\mathbb{R}_t; H^s(\mathbb{T}))$.

The space $Y_s^T = \{f|_{[-T,T]} : f \in Y_s\}$ with the norm

$$\|f\|_{Y_s^T} = \inf \{ \|g\|_{Y_s} : g|_{[-T,T]} = f \text{ and } g \in Y_s \},$$

satisfies $Y_s^T \subset C([-T, T]; H^s(\mathbb{T}))$.

Now, we are in position to state the local result.

Theorem 1.6. *Suppose that $\frac{q}{3} \notin \mathbb{Z}$. If $s \geq \frac{1}{2}$ and $\vec{u}_0 = (u_0, w_0) \in H^s(\mathbb{T}) \times H^s(\mathbb{T})$, then there exist $T(\|\vec{u}_0\|_{H^s}) > 0$ and a unique solution $\vec{u} = (u, w)$ to the IVP (1.1) in the case $(\beta + \mu) = \beta\sigma_\beta$ satisfying*

$$\vec{u} \in C([-T, T] : H^s(\mathbb{T}) \times H^s(\mathbb{T})), \quad \vec{u} \in Y_s \times Y_s,$$

where $c_0 = (\beta + \mu)\|\vec{u}_0\|_{L_x^2}^2$ in the definition of Y_s .

For each $T' \in (0, T)$, there exists $\epsilon > 0$ such that the map $\vec{v}_0 \mapsto \vec{v}$ is Lipschitz continuous from

$$\{\vec{v}_0 \in H^s(\mathbb{T}) \times H^s(\mathbb{T}) : \|\vec{v}_0 - \vec{u}_0\|_{H^s} < \epsilon\}$$

to

$$\left\{ \vec{v} : \|\vec{v} - \vec{u}\|_{L_{T'}^\infty H^s} + \|\Psi_{T'}(\vec{v} - \vec{u})\|_{Y_s} < \infty \right\}.$$

We notice that the solution flow of (1.1) is invariant by the following quantities in the case $\sigma_\alpha = \sigma_\beta = \sigma_\mu = 1$.

$$I_1(u, w) = \|u\|_{L^2(\Omega)}^2 + \|w\|_{L^2(\Omega)}^2 = I_1(u_0, w_0) \quad (1.4)$$

and

$$\begin{aligned} I_2(u, w) &= i(-3\gamma\alpha + \beta q + 2\mu q) \int_{\Omega} (u\bar{u}_x + w\bar{w}_x) dx + \frac{3}{2}\gamma \int_{\Omega} (|u_x|^2 + |w_x|^2) dx \\ &\quad + \frac{1}{2}(\beta + 2\mu) \int_{\Omega} (|u|^4 + |w|^4) dx + (\beta + 2\mu) \int_{\Omega} |u|^2 |w|^2 dx \\ &= I_2(u_0, w_0) \end{aligned} \quad (1.5)$$

for either $\Omega = \mathbb{R}$ or \mathbb{T} , see [4]. These quantities allow us to extend our local result globally. This is what is contained in the next result.

Before stating our main results, let us define the notation that will be used throughout this work.

Definition 1.7. Let $\mathcal{P} = \mathcal{C}^\infty(\mathbb{T}) = \{g : \mathbb{R} \rightarrow \mathbb{C}; g \in \mathcal{C}^\infty \text{ periodic with period } 2\pi\}$. \mathcal{P}' (dual of \mathcal{P}) is the collection of all linear functionals from \mathcal{P} to \mathbb{C} . \mathcal{P}' is periodic distributions. If $g \in \mathcal{P}'$ denote the value of g in φ by $g(\varphi) = \langle g, \varphi \rangle$. Consider the functions $\theta_n(x) = e^{inx}$, $n \in \mathbb{Z}$ and $x \in \mathbb{R}$. The Fourier transform $g \in \mathcal{P}'$ is the function $\hat{g} : \mathbb{Z} \rightarrow \mathbb{C}$ defined by $\hat{g}(n) = \langle g, \theta_{-n} \rangle$. If g is periodic of period 2π , for instance, $g \in L^1(\mathbb{T})$ then

$$\hat{g}(n) = \int_{\mathbb{T}} e^{-inx} g(x) dx = \frac{1}{2\pi} \int_0^{2\pi} e^{-inx} g(x) dx,$$

where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ represents the one dimensional torus.

Definition 1.8. Let $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{T})$ is the set of all $g \in \mathcal{P}'$ such that

$$\|g\|_{H^s(\mathbb{T})} = \left(2\pi \sum_{n \in \mathbb{Z}} (1 + |n|^2)^s |\hat{g}(n)|^2 \right)^{\frac{1}{2}} < \infty.$$

In this work we assume that g is periodic of period 2π .

Definition 1.9. We denote by \tilde{f} the Fourier transform of f in relation to space-time variables

$$\tilde{f}(n, \tau) = \int_{\mathbb{R}} \int_{\mathbb{T}} e^{-i(xn + t\tau)} f(x, t) dx dt.$$

The inverse Fourier transform is given by

$$f(x, t) = \sum_{n \in \mathbb{Z}} e^{inx} \int_{\mathbb{R}} e^{it\tau} \tilde{f}(n, \tau) d\tau.$$

The Fourier transform of fgh , where $f = f(x, t)$, $g = g(x, t)$ and $h = h(x, t)$ are periodic functions with respect to the x variable obtained as

$$\widehat{fgh}(n, \tau) = (\tilde{f} * \tilde{g} * \tilde{h})(n, \tau) = \sum_{n_1 \in \mathbb{Z}} \sum_{n_2 \in \mathbb{Z}} \iint_{\mathbb{R}^2} \tilde{f}(n_1, \tau_1) \tilde{g}(n_2, \tau_2) \tilde{h}(n - n_1 - n_2, \tau - \tau_1 - \tau_2) d\tau_1 d\tau_2.$$

Definition 1.10. Let \mathcal{V} be the space of functions f such that

- (i) $f : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{C}$,
- (ii) $f(x, \cdot) \in \mathcal{S}(\mathbb{R})$ for each $x \in \mathbb{T}$,
- (iii) $f(\cdot, t) \in C^\infty(\mathbb{T})$ for each $t \in \mathbb{R}$.

We define the space $X_{s,b}$ associated to operator $\partial_t - i\frac{q}{2}\partial_x^2 + \partial_x^3 + c_0\partial_x$ as the completion of \mathcal{V} with respect to the following norm

$$\|f\|_{X_{s,b}} := \|f\|_{(1,s,b)} = \|\langle n \rangle^s \langle \tau - n^3 + \frac{q}{2}n^2 + c_0n \rangle^b \tilde{f}(n, \tau)\|_{l_n^2 L_\tau^2},$$

where $\langle n \rangle = (1 + |n|^2)^{\frac{1}{2}}$ and $s, b, c_0 \in \mathbb{R}$.

The space $Z_{s,b}$ is the completion of \mathcal{V} with respect to the norm

$$\|f\|_{Z_{s,b}} := \|f\|_{(2,s,b)} = \|\langle n \rangle^s \langle \tau - n^3 + \frac{q}{2}n^2 + c_0n \rangle^b \tilde{f}(n, \tau)\|_{l_n^2 L_\tau^1},$$

and we consider the space $Y_s = X_{s, \frac{1}{2}} \cap Z_{s,0}$ with the norm

$$\|f\|_{Y_s} = \|f\|_{(1,s,\frac{1}{2})} + \|f\|_{(2,s,0)}.$$

Remark 1.11. Note that for $b > \frac{1}{2}$ we have that $X_{s,b} \subset C(\mathbb{R}_t; H^s(\mathbb{T}))$ and for $b = 0$, we have $Z_{s,0} \subset C(\mathbb{R}_t; H^s(\mathbb{T}))$ and $Y_s \subset C(\mathbb{R}_t; H^s(\mathbb{T}))$.

The space $Y_s^T = \{f|_{[-T,T]} : f \in Y_s\}$ with the norm

$$\|f\|_{Y_s^T} = \inf \{ \|g\|_{Y_s} : g|_{[-T,T]} = f \text{ and } g \in Y_s \},$$

satisfies $Y_s^T \subset C([-T, T]; H^s(\mathbb{T}))$.

Now, we are in position to state the main results of this work.

Theorem 1.12. Suppose that $\frac{q}{3} \notin \mathbb{Z}$. If $s \geq \frac{1}{2}$ and $\vec{u}_0 = (u_0, w_0) \in H^s(\mathbb{T}) \times H^s(\mathbb{T})$, then there exist $T(\|\vec{u}_0\|_{H^s}) > 0$ and a unique solution $\vec{u} = (u, w)$ to the IVP (1.1) in the case $(\beta + \mu) = \beta\sigma_\beta$ satisfying

$$\vec{u} \in C([-T, T] : H^s(\mathbb{T}) \times H^s(\mathbb{T})), \quad \vec{u} \in Y_s \times Y_s,$$

where $c_0 = (\beta + \mu)\|\vec{u}_0\|_{L_x^2}^2$ in the definition of Y_s .

For each $T' \in (0, T)$, there exists $\epsilon > 0$ such that the map $\vec{v}_0 \mapsto \vec{v}$ is Lipschitz continuous from

$$\{\vec{v}_0 \in H^s(\mathbb{T}) \times H^s(\mathbb{T}) : \|\vec{v}_0 - \vec{u}_0\|_{H^s} < \epsilon\}$$

to

$$\left\{ \vec{v} : \|\vec{v} - \vec{u}\|_{L_{T'}^\infty H^s} + \|\Psi_{T'}(\vec{v} - \vec{u})\|_{Y_s} < \infty \right\}.$$

Combining the local well-posedness result with the conservation laws the following global result follows.

Theorem 1.13. *Let $\vec{u}_0 = (u_0, w_0) \in H^1(\mathbb{T}) \times H^1(\mathbb{T})$. Then there exists a unique solution $\vec{u} = (u, w)$ to the problem (1.1) with $\sigma_\alpha = \sigma_\beta = \sigma_\mu = 1$ and $\mu = 0$ satisfying*

$$(u, w) \in C(\mathbb{R}; H^1(\mathbb{T}) \times H^1(\mathbb{T})).$$

This work is organized as follows. In second section we will list a series of estimates in the spaces defined on Definition 1.4 that will be needed in the proof of Theorem 1.12. In the third section we establish local well-posedness for the periodic initial value problem associated to (1.1) for data in $H^s(\mathbb{T}) \times H^s(\mathbb{T})$, $s \geq \frac{1}{2}$ and the fourth section is dedicated to global result. We finish the paper with some comments about future work.

2. PRELIMINARY ESTIMATES

To prove our periodic results we use the spaces introduced by Bourgain in [3], the contraction principle and also the properties of the solutions to the linear problem

$$\begin{cases} \partial_t u - i\frac{\gamma}{2}\partial_x^2 u + \partial_x^3 u + c_0 \partial_x u = 0, & x \in \mathbb{T}, t > 0, \\ \partial_t w - i\frac{\gamma}{2}\partial_x^2 w + \partial_x^3 w + c_0 \partial_x w = 0, \\ u(x, 0) = u_0(x) \text{ and } w(x, 0) = w_0(x). \end{cases} \quad (2.1)$$

This linear system differs from the one used in [4] because of the terms containing $c_0 \partial_x u$ and $c_0 \partial_x w$, where the constant $c_0 = (\beta + \mu) [\|u\|_{L_x^2}^2 + \|w\|_{L_x^2}^2]$.

Remark 2.1. *The problem (2.1) is a particular case of (1.1) with $\gamma = 2$. So, we assume $\gamma = 2$ without loss of generality because if $\gamma \neq 2$, it is enough to consider the change of variable $v_1(x, t) = u(\theta x, t)$ and $v_2(x, t) = w(\theta x, t)$, where $\theta = \sqrt[3]{\frac{\gamma}{2}}$, obtaining $u(x, t) = v_1(\theta^{-1}x, t)$, $w(x, t) = v_2(\theta^{-1}x, t)$, $\frac{\gamma}{2}\partial_x^3 u = \partial_x^3 v_1$ and $\frac{\gamma}{2}\partial_x^3 w = \partial_x^3 v_2$.*

We note that $\|u(t)\|_{L_x^2}^2 + \|w(t)\|_{L_x^2}^2$ is a conserved quantity, see (1.4). Therefore the constant c_0 is independent of t . In what follows this constant is used as the number c_0 in the Definition 1.4. It plays important role to get the bounds we need.

The solution of (2.1) is given by the unitary group $\{W_p(t)\}_{t \in \mathbb{R}}$ in $H^s(\mathbb{T}) \times H^s(\mathbb{T})$, defined as

$$\vec{u}(x, t) = W_p(t) \vec{u}_0 = (S_p(t)u_0, S_p(t)w_0), \quad (2.2)$$

where, the subscript p only means "periodic", and

$$S_p(t)u_0 = \sum_{n \in \mathbb{Z}} e^{inx} e^{it\phi(n)} \widehat{u}_0(n),$$

and $\phi(n) = n^3 - \frac{\gamma}{2}n^2 - c_0 n$.

Let $q_\pm(n, \tau) = \tau - n^3 \pm \frac{\gamma}{2}n^2 + c_0 n$, then we obtain the following equalities,

$$\langle q_-(n, \tau) \rangle^b |\widetilde{u}(n, \tau)| = \langle q_+(-n, -\tau) \rangle^b |\widetilde{u}(-n, -\tau)|, \quad (2.3)$$

$$\|f\|_{(1,s,b)} = \|\langle n \rangle^s \langle q_+(n, \tau) \rangle^b \widetilde{f}(n, \tau)\|_{l_n^2 L_\tau^2}. \quad (2.4)$$

The main linear estimates are the following.

Lemma 2.1. *For $s \in \mathbb{R}$ we have*

$$\|\Psi(t)W_p(t)\vec{u}_0\|_{(1,s,\frac{1}{2})} \leq c\|\vec{u}_0\|_{H^s \times H^s},$$

$$\|\Psi(t)W_p(t)\vec{u}_0\|_{(2,s)} \leq c\|\vec{u}_0\|_{H^s \times H^s}.$$

Therefore

$$\|\Psi(t)W_p(t)\vec{u}_0\|_{Y_s \times Y_s} \leq c\|\vec{u}_0\|_{H^s \times H^s}, \quad (2.5)$$

where $\Psi(t)W_p(t)\vec{u}_0$ is given by

$$\Psi(t)W_p(t)\vec{u}_0 = (\psi(t)S_p(t)u_0, \psi(t)S_p(t)w_0)$$

and ψ denotes a cut-off function satisfying $\psi = 1$ in $[-1, 1]$, $\psi \in C_0^\infty$ and $\text{supp } \psi \subseteq (-2, 2)$.

Proof. Taking in account the Definition 1.4, to obtain (2.5) it is enough to estimate

$$\begin{aligned} \|\psi(t)S_p(t)u_0\|_{(1,s,\frac{1}{2})}^2 &= \|\langle n \rangle^s \langle q_+(n, \tau) \rangle^{\frac{1}{2}} \widehat{\psi}(q_+(n, \tau)) \widehat{u}_0(n)\|_{l_n^2 L_\tau^2}^2 \\ &= \sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} |\widehat{u}_0(n)|^2 \int_{\mathbb{R}} \langle q_+(n, \tau) \rangle^1 |\widehat{\psi}(q_+(n, \tau))|^2 d\tau \\ &\leq c\|u_0\|_{H^s}^2 \|\psi\|_{H^{1/2}}^2 \leq c\|u_0\|_{H^s}^2 \end{aligned} \quad (2.6)$$

and

$$\begin{aligned} \|\psi(t)S_p(t)u_0\|_{(2,s,0)}^2 &= \sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} |\widehat{u}_0(n)|^2 \left(\int_{\mathbb{R}} |\widehat{\psi}(q_+(n, \tau))| d\tau \right)^2 \\ &\leq c\|u_0\|_{H^s}^2. \end{aligned} \quad (2.7)$$

Details of the computations are found in [11]. \square

3. PROOF OF THEOREM 1.6

The result in this section requires a new set of computations, so we will present it in more detailed setting. To simplify the notation we write (1.1) as

$$\begin{cases} \partial_t \vec{u} - \frac{q}{2} i \partial_x^2 \vec{u} + \partial_x^3 \vec{u} + \mathbf{c}_0 \partial_x \vec{u} = G(\vec{u}), \\ \vec{u}(x, 0) = \vec{u}_0 \in H^s(\mathbb{T}) \times H^s(\mathbb{T}), \end{cases} \quad (3.1)$$

where

$$G(\vec{u}) = (G_1(u, w), G_1(w, u)),$$

with

$$\begin{aligned} G_1(u, w) &= (\beta + \mu) \left[|u|^2 - \|u\|_{L_x^2}^2 - \|w\|_{L_x^2}^2 \right] \partial_x u + \beta \sigma_\beta |w|^2 \partial_x u \\ &\quad + \mu u^2 \partial_x \bar{u} + \mu \sigma_\mu u \partial_x (|w|^2) - i \alpha u (|u|^2 + \sigma_\alpha |w|^2). \end{aligned} \quad (3.2)$$

The integral equation associated to (3.1) is

$$\Phi_{\vec{u}_0}(\vec{u}) = W_p(t) \vec{u}_0 - \int_0^t W_p(t-t') G(\vec{u})(t') dt', \quad (3.3)$$

where $W_p(t)$ is defined in (2.2).

Considering the cut-off function $\psi \in C_0^\infty$ defined on Lemma 2.1 we have the following estimate.

Lemma 3.1. *For $s \in \mathbb{R}$ we have*

$$\begin{aligned} \|\Psi(t) \int_0^t W_p(t-t') G(\vec{u})(t') dt'\|_{Y_s \times Y_s} &\leq c\|G_1(u, w)\|_{(1,s,-\frac{1}{2})} \\ &\quad + c\|G_1(w, u)\|_{(1,s,-\frac{1}{2})} + c \left(\sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \left(\int_{-\infty}^{+\infty} \frac{\widetilde{G_1(u, w)}(n, \tau)}{\langle q_+(n, \tau) \rangle} d\tau \right)^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (3.4)$$

Proof. To obtain (3.4) it is sufficient to estimate

$$\begin{aligned} \|\psi(t) \int_0^t S_p(t-t') G_1(u, w)(t') dt'\|_{Y_s} &\leq c \|G_1(u, w)\|_{(1, s, -\frac{1}{2})} \\ &+ c \left(\sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \left(\int_{-\infty}^{+\infty} \frac{\widetilde{G_1(u, w)}(n, \tau)}{\langle q_+(n, \tau) \rangle} d\tau \right)^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (3.5)$$

From the relation $\int_0^t h(t') dt' = \int_{-\infty}^{+\infty} \frac{e^{it\lambda}-1}{i\lambda} \widehat{h}(\lambda) d\lambda$ and $G_1(x, t) = G_1(u, w)(x, t)$, we obtain

$$\begin{aligned} \psi(t) \int_0^t S_p(t-t') G_1(x, t') dt' &= \\ \psi(t) \sum_{n \in \mathbb{Z}} e^{inx} \int_{-\infty}^{+\infty} \left(\frac{e^{it(q_+(n, \lambda))} - 1}{i(q_+(n, \lambda))} \right) e^{it(n^3 - \frac{g}{2}n^2 - c_0n)} \widetilde{G}_1(n, \lambda) d\lambda. \end{aligned} \quad (3.6)$$

Let $\varphi \in C_0^\infty$ be another function such that $\varphi \equiv 1$ in $[-B, B]$, $\text{supp } \varphi \subseteq (-2B, 2B)$, where $B < \frac{1}{100}$, say. Using φ , we can write (3.6) as

$$\begin{aligned} &\psi(t) \sum_{n \in \mathbb{Z}} e^{inx} \int_{-\infty}^{+\infty} \left(\frac{e^{it(q_+(n, \lambda))} - 1}{i(q_+(n, \lambda))} \right) e^{it\phi(n)} \varphi(q_+(n, \lambda)) \widetilde{G}_1(n, \lambda) d\lambda \\ &+ \psi(t) \sum_{n \in \mathbb{Z}} e^{inx} \int_{-\infty}^{+\infty} \left(\frac{e^{it(q_+(n, \lambda))} - 1}{i(q_+(n, \lambda))} \right) e^{it\phi(n)} (1 - \varphi(q_+(n, \lambda))) \widetilde{G}_1(n, \lambda) d\lambda \\ &= J_1(x, t) + J_2(x, t). \end{aligned}$$

That is,

$$\begin{aligned} J_1(x, t) &= \psi(t) \sum_{n \in \mathbb{Z}} e^{inx} e^{it\phi(n)} \times \\ &\int_{-\infty}^{+\infty} \sum_{k \geq 1} \left(\frac{(q_+(n, \lambda))^{k-1} e^{it(q_+(n, \lambda))}}{k!} i^{k-1} t^k \right) \varphi(q_+(n, \lambda)) \widetilde{G}_1(n, \lambda) d\lambda \\ J_2(x, t) &= \psi(t) \sum_{n \in \mathbb{Z}} e^{inx} e^{it\phi(n)} \int_{-\infty}^{+\infty} \left(\frac{e^{it(q_+(n, \lambda))}}{i(q_+(n, \lambda))} \right) (1 - \varphi(q_+(n, \lambda))) \widetilde{G}_1(n, \lambda) d\lambda \\ &- \psi(t) \sum_{n \in \mathbb{Z}} e^{inx} e^{it\phi(n)} \int_{-\infty}^{+\infty} \left(\frac{1 - \varphi(q_+(n, \lambda))}{i(q_+(n, \lambda))} \right) \widetilde{G}_1(n, \lambda) d\lambda \\ &= J_2^1(x, t) + J_2^2(x, t). \end{aligned} \quad (3.7)$$

Therefore, $\|\psi(t) \int_0^t S_p(t-t') G_1(u, w)(t') dt'\|_{Y_s} \leq \|J_1\|_{Y_s} + \|J_2^1\|_{Y_s} + \|J_2^2\|_{Y_s}$.

Now, we compute the norms of J_1 , J_2^1 and J_2^2

$$\begin{aligned} \|J_1\|_{(1, s, \frac{1}{2})} &= \|\langle n \rangle^s \langle q_+(n, \tau) \rangle^{\frac{1}{2}} \widetilde{J}_1(n, \tau)\|_{l_n^2 L_\tau^2} \\ &= \|\langle n \rangle^s \langle q_+(n, \tau) \rangle^{\frac{1}{2}} \sum_{k \geq 1} \frac{h_k(n)}{k!} \widehat{\psi}_k(q_+(n, \tau))\|_{l_n^2 L_\tau^2}, \end{aligned} \quad (3.8)$$

where

$$h_k(n) = \int_{-\infty}^{+\infty} i^{k-1} (q_+(n, \lambda))^{k-1} \varphi(q_+(n, \lambda)) \widetilde{G}_1(n, \lambda) d\lambda$$

and

$$\widehat{\psi}_k(q_+(n, \tau)) = \int_{-\infty}^{+\infty} e^{-it(q_+(n, \tau))} \psi(t) t^k dt.$$

Using the properties of φ we estimate

$$|h_k(n)| \leq c \|\langle q_+(n, \lambda) \rangle^{-\frac{1}{2}} \widetilde{G}_1(n, \lambda)\|_{L_\lambda^2} = L(n). \quad (3.9)$$

After an integration by parts and using properties of ψ , we have

$$|h_k(n)| \leq c \frac{k^2 + k + 1}{|q_+(n, \tau)|^2}. \quad (3.10)$$

On the other hand, we have

$$|\widehat{\psi}_k(q_+(n, \tau))| \leq \|t^k \psi\|_{L^1} \leq c. \quad (3.11)$$

It follows from (3.9)-(3.11) that

$$|\widehat{\psi}_k(q_+(n, \tau))| (1 + |q_+(n, \tau)|^2) \leq c(k^2 + k + 1). \quad (3.12)$$

From (3.8)-(3.12) we obtain

$$\begin{aligned} \|J_1\|_{(1, s, \frac{1}{2})} &\leq \left(\sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \int_{-\infty}^{+\infty} \langle q_+(n, \tau) \rangle \left(\sum_{k \geq 1} \frac{|h_k(n)|}{k!} |\widehat{\psi}_k(q_+(n, \tau))| \right)^2 d\tau \right)^{\frac{1}{2}} \\ &\leq c \left(\sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} |L(n)|^2 \int_{-\infty}^{+\infty} \frac{1}{\langle q_+(n, \tau) \rangle^3} \left(\sum_{k \geq 1} \frac{k^2 + k + 1}{k!} \right)^2 d\tau \right)^{\frac{1}{2}} \\ &\leq c \left(\sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} |L(n)|^2 \right)^{\frac{1}{2}} = c \|G_1(u, w)\|_{(1, s, -\frac{1}{2})}. \end{aligned} \quad (3.13)$$

The following estimate follows from Young's inequality and $|\frac{1-\varphi(x)}{x}| \leq c_1(B) \langle x \rangle^{-\frac{1}{2}}$.

$$\begin{aligned} \|J_2^1\|_{(1, s, \frac{1}{2})} &= \|\langle n \rangle^s \langle q_+(n, \tau) \rangle^{\frac{1}{2}} \left(\widehat{\psi}(\cdot) * \frac{(1 - \varphi(q_+(n, \cdot))) \widetilde{G}_1(n, \cdot)}{q_+(n, \cdot)} \right)(\tau)\|_{l_n^2 L_\tau^2} \\ &\leq \left(\sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \|\langle q_+(n, \tau) \rangle \widehat{\psi}(\tau)\|_{L_\tau^1}^2 \left\| \frac{(1 - \varphi(q_+(n, \tau))) \widetilde{G}_1(n, \tau)}{q_+(n, \tau)} \right\|_{L_\tau^2}^2 \right)^{\frac{1}{2}} \\ &\leq c \left(\sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \left\| \frac{(1 - \varphi(q_+(n, \tau))) \widetilde{G}_1(n, \tau)}{q_+(n, \tau)} \right\|_{L_\tau^2}^2 \right)^{\frac{1}{2}} \\ &\leq c \left(\sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \|\langle q_+(n, \lambda) \rangle^{-\frac{1}{2}} \widetilde{G}_1(n, \tau)\|_{L_\tau^2}^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (3.14)$$

To estimate $\|J_2^2\|_{(1, s, \frac{1}{2})}$ note that

$$J_2^2 = -\psi(t) \sum_{n \in \mathbb{Z}} e^{inx} e^{it(n^3 - \frac{g}{2}n^2 - c_0 n)} \widehat{r}(n) = -\psi(t) S(t) r(x),$$

where

$$\widehat{r}(n) = \int_{-\infty}^{+\infty} \left(\frac{1 - \varphi(q_+(n, \lambda))}{i(q_+(n, \lambda))} \right) \widetilde{G}_1.$$

Then (2.6) and the inequality $|\frac{1-\varphi(x)}{x}| \leq \frac{c(B)}{\langle x \rangle^1}$ imply

$$\begin{aligned} \|J_2^2\|_{(1,s,\frac{1}{2})} &= \|\psi(t)S_p(t)r\|_{(1,s,\frac{1}{2})} \leq \|r\|_{H^s} \\ &\leq c \left(\sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \left| \int_{-\infty}^{+\infty} \left(\frac{1-\varphi(q_+(n,\lambda))}{i(q_+(n,\lambda))} \right) \widetilde{G}_1(n,\lambda) d\lambda \right|^2 \right)^{\frac{1}{2}} \\ &\leq c \left(\sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \left| \int_{-\infty}^{+\infty} \frac{\widetilde{G}_1(n,\lambda)}{\langle q_+(n,\lambda) \rangle} d\lambda \right|^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (3.15)$$

Using the definition of $h_k(n)$ it is straightforward to obtain

$$\|h_k(n)\| \leq \|\chi_{[-1,1]}(q_+(n,\lambda)) \widetilde{G}_1(n,\lambda)\|_{L_\lambda^1} \leq c \left\| \frac{\widetilde{G}_1(n,\lambda)}{\langle q_+(n,\lambda) \rangle} \right\|_{L_\lambda^1}.$$

Then,

$$\begin{aligned} \|J_1\|_{(2,s,0)} &= \|\langle n \rangle^s \int_{-\infty}^{+\infty} e^{-it(q_+(n,\tau))} \psi(t) \sum_{k \geq 1} \frac{t^k}{k!} h_k(n) dt\|_{L_n^2 L_\tau^1} \\ &\leq c \left(\sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \left(\int_{-\infty}^{+\infty} \sum_{k \geq 1} \frac{|h_k(n)|}{k!} |\widehat{\psi}_k(q_+(n,\tau))| d\tau \right)^2 \right)^{\frac{1}{2}} \\ &\leq c \left(\sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \left\| \frac{\widetilde{G}_1(n,\lambda)}{\langle q_+(n,\lambda) \rangle} \right\|_{L_\lambda^1}^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (3.16)$$

Proceeding analogously to (3.14) and (3.15), we can show that

$$\|J_2^1\|_{(2,s,0)} + \|J_2^2\|_{(2,s,0)} \leq c \left(\sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \left\| \frac{\widetilde{G}_1(n,\tau)}{\langle q_+(n,\tau) \rangle} \right\|_{L_\tau^1}^2 \right)^{\frac{1}{2}},$$

which finishes the proof of the lemma. \square

Let $n, n_1, n_2 \in \mathbb{Z}$, $\tau, \tau_1, \tau_2 \in \mathbb{R}$, and set $n_3 = n - n_1 - n_2$ and $\tau_3 = \tau - \tau_1 - \tau_2$, then

$$\begin{aligned} q_+(n, \tau) - q_+(n_1, \tau_1) - q_-(n_2, \tau_2) - q_+(n_3, \tau_3) \\ = -3(n_1 + n_2)(n - n_1)(n - n_2 - \frac{q}{3}). \end{aligned} \quad (3.17)$$

Therefore,

$$\max\{|q_+(n, \tau)|, |q_+(n_1, \tau_1)|, |q_-(n_2, \tau_2)|, |q_+(n_3, \tau_3)|\} \geq \frac{3}{4}|n_1 + n_2||n - n_1||n - n_2 - \frac{q}{3}|.$$

Now, we define $M_1(n, n_1, n_2) = M_1$, $L(n, n_1, n_2) = L$ and $M_2(n, n_1, n_2) = M_2$ as

$$\begin{aligned} M_1(n, n_1, n_2) &= \max\{|n - n_1|, |n_1 + n_2|, |n - n_2|\}, \\ M_2(n, n_1, n_2) &= \min\{|n - n_1|, |n_1 + n_2|, |n - n_2|\}, \end{aligned}$$

and

$$L(n, n_1, n_2) = \begin{cases} |n - n_1|, & \text{if } (|n - n_1| - |n_1 + n_2|)(|n - n_1| - |n - n_2|) \leq 0, \\ |n_1 + n_2|, & \text{if } (|n_1 + n_2| - |n - n_1|)(|n_1 + n_2| - |n - n_2|) \leq 0, \\ |n - n_2|, & \text{if } (|n - n_2| - |n - n_1|)(|n - n_2| - |n_1 + n_2|) \leq 0. \end{cases}$$

Note that $M_2 \leq L \leq M_1$. The following inequalities will be useful to prove the next lemma. The nonlinear term $G_1(u, w)$ defined in (3.2) restricted to $(\beta + \mu) = \beta\sigma_\beta$, writes

$$\begin{aligned} G_1(u, w) &= (\beta + \mu) [|u|^2 - \|u\|_{L_x^2}^2 + |w|^2 - \|w\|_{L_x^2}^2] \partial_x u \\ &\quad + (\mu u^2 \partial_x \bar{u} + \mu \sigma_\mu u w \partial_x \bar{w}) + \mu \sigma_\mu u \bar{w} \partial_x w - i\alpha u (|u|^2 + \sigma_\alpha |w|^2) \\ &= G_{11}(u, w) + G_{12}(u, w) + G_{13}(u, w) + G_{14}(u, w). \end{aligned} \quad (3.18)$$

Denote by $G_{1j} = G_{1j}(u, w)$, for $j = 1, \dots, 4$. Next, we compute $\widetilde{G_{1j}}(n, \tau)$. To do so, note that using Parseval's identity,

$$\begin{aligned} \widetilde{|u|^2 u_x}(n, \tau) &= \int_{\mathbb{R}} e^{-it\tau} \left(\widehat{u} * \widehat{u} * \widehat{u_x} \right) dt = \int_{\mathbb{R}} e^{-it\tau} \sum_{n_1 \in \mathbb{Z}} in_1 \widehat{u}(n_1) \left(\widehat{u} * \widehat{u} \right) (n - n_1) dt \\ &= \int_{\mathbb{R}} e^{-it\tau} \sum_{n_1 \neq n} \sum_{n_2 \in \mathbb{Z}} in_1 \widehat{u}(n_1) \widehat{u}(n_2) \widehat{u}(n_3) dt + \int_{\mathbb{R}} e^{-it\tau} in \widehat{u}(n) \|u\|_{L_x^2}^2 \\ &= \int_{\mathbb{R}} e^{-it\tau} \sum_{n_1 \neq n} \sum_{n_2 \neq -n_1} in_1 \widehat{u}(n_1) \widehat{u}(n_2) \widehat{u}(n_3) dt \\ &\quad + \int_{\mathbb{R}} e^{-it\tau} \sum_{n_1 \in \mathbb{Z}} in_1 \widehat{u}(n_1) \widehat{u}(-n_1) \widehat{u}(n) dt - \int_{\mathbb{R}} e^{-it\tau} in \widehat{u}(n) \widehat{u}(-n) \widehat{u}(n) dt \\ &\quad + \int_{\mathbb{R}} e^{-it\tau} in \widehat{u}(n) \|u\|_{L_x^2}^2 dt. \end{aligned}$$

Therefore,

$$\begin{aligned} \widetilde{|u|^2 u_x}(n, \tau) &= \sum_{(n_1, n_2) \in H_n^1} in_1 \iint_{\mathbb{R}^2} \widetilde{u}(n_1, \tau_1) \widetilde{u}(n_2, \tau_2) \widetilde{u}(n_3, \tau_3) d\tau_1 d\tau_2 \\ &\quad + \sum_{n_1 \in \mathbb{Z}} in_1 \iint_{\mathbb{R}^2} \widetilde{u}(n_1, \tau_1) \widetilde{u}(-n_1, \tau_2) \widetilde{u}(n, \tau_3) d\tau_1 d\tau_2 \\ &\quad - \iint_{\mathbb{R}^2} in \widetilde{u}(n, \tau_1) \widetilde{u}(-n, \tau_2) \widetilde{u}(n, \tau_3) d\tau_1 d\tau_2 \\ &\quad + \int_{\mathbb{R}} e^{-it\tau} in \widehat{u}(n) \|u\|_{L_x^2}^2 dt, \end{aligned} \quad (3.19)$$

where $H_n^1 = \{(n_1, n_2) \in \mathbb{Z}^2 : n - n_1 \neq 0, n_1 + n_2 \neq 0\}$. The term $\widetilde{|w|^2 u_x}(n, \tau)$ is computed analogously.

We write $\widetilde{|u|^2 u_x} = A + \widetilde{|u|_{L_x^2}^2 u_x}$ and $\widetilde{|w|^2 u_x} = B + \widetilde{|w|_{L_x^2}^2 u_x}$. Therefore $\widetilde{G}_{11}(n, \tau)$ will be defined as $A(n, \tau) + B(n, \tau)$. Explicitly we have that $\widetilde{G}_{11}(n, \tau)$:

$$\begin{aligned}
\widetilde{G}_{11}(n, \tau) &= c_1(\beta, \mu) \sum_{(n_1, n_2) \in H_n^1} in_1 \iint_{\mathbb{R}^2} \widetilde{u}(n_1, \tau_1) \widetilde{u}(n_2, \tau_2) \widetilde{u}(n_3, \tau_3) d\tau_1 d\tau_2 \\
&+ c_1(\beta, \mu) \sum_{(n_1, n_2) \in H_n^1} in_1 \iint_{\mathbb{R}^2} \widetilde{u}(n_1, \tau_1) \widetilde{w}(n_2, \tau_2) \widetilde{w}(n_3, \tau_3) d\tau_1 d\tau_2 \\
&+ c_1(\beta, \mu) \sum_{n_1 \in \mathbb{Z}} in_1 \iint_{\mathbb{R}^2} \widetilde{u}(n_1, \tau_1) \widetilde{u}(-n_1, \tau_2) \widetilde{u}(n, \tau_3) d\tau_1 d\tau_2 \\
&+ c_1(\beta, \mu) \sum_{n_1 \in \mathbb{Z}} in_1 \iint_{\mathbb{R}^2} \widetilde{u}(n_1, \tau_1) \widetilde{w}(-n_1, \tau_2) \widetilde{w}(n, \tau_3) d\tau_1 d\tau_2 \\
&- c_1(\beta, \mu) \iint_{\mathbb{R}^2} in \widetilde{u}(n, \tau_1) \widetilde{u}(-n, \tau_2) \widetilde{u}(n, \tau_3) d\tau_1 d\tau_2 \\
&- c_1(\beta, \mu) \iint_{\mathbb{R}^2} in \widetilde{u}(n, \tau_1) \widetilde{w}(-n, \tau_2) \widetilde{w}(n, \tau_3) d\tau_1 d\tau_2 \\
&= R_1(n, \tau) + R_2(n, \tau) + R_3(n, \tau) + R_4(n, \tau) + R_5(n, \tau) + R_6(n, \tau).
\end{aligned} \tag{3.20}$$

Remark 3.1. The computation of $\widetilde{G}_{13}(n, \tau)$ and $\widetilde{G}_{12}(n, \tau)$ are similar to $\widetilde{G}_{11}(n, \tau)$.

We have for $\widetilde{G}_{14}(n, \tau)$:

$$\begin{aligned}
\widetilde{G}_{14}(n, \tau) &= c_4(\alpha) \sum_{(n_1, n_2) \in H_n^1} \iint_{\mathbb{R}^2} \widetilde{u}(n_1, \tau_1) \widetilde{u}(n_2, \tau_2) \widetilde{u}(n_3, \tau_3) d\tau_1 d\tau_2 \\
&+ c_5(\alpha, \sigma_\alpha) \sum_{(n_1, n_2) \in H_n^1} \iint_{\mathbb{R}^2} \widetilde{u}(n_1, \tau_1) \widetilde{w}(n_2, \tau_2) \widetilde{w}(n_3, \tau_3) d\tau_1 d\tau_2 \\
&+ c_6(\alpha) \iint_{\mathbb{R}^2} \|\widetilde{u}(\tau_1)\|_{l_n^2}^2 \|\widetilde{u}(\tau_2)\|_{l_n^2}^2 \widetilde{u}(n, \tau_3) d\tau_1 d\tau_2 \\
&+ c_7(\alpha, \sigma_\alpha) \iint_{\mathbb{R}^2} \|\widetilde{w}(\tau_1)\|_{l_n^2}^2 \|\widetilde{w}(\tau_2)\|_{l_n^2}^2 \widetilde{u}(n, \tau_3) d\tau_1 d\tau_2 \\
&+ c_8(\alpha) \iint_{\mathbb{R}^2} \widetilde{u}(n, \tau_1) \widetilde{u}(-n, \tau_2) \widetilde{u}(n, \tau_3) d\tau_1 d\tau_2 \\
&+ c_9(\alpha) \iint_{\mathbb{R}^2} \widetilde{w}(n, \tau_1) \widetilde{w}(-n, \tau_2) \widetilde{u}(n, \tau_3) d\tau_1 d\tau_2.
\end{aligned} \tag{3.21}$$

Lemma 3.2. Assume $\frac{q}{3}$ is not an integer and $0 < \theta < \frac{1}{12}$. For $s \geq \frac{1}{2}$ there exists $c > 0$ such that

$$\left(\sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \left(\int_{-\infty}^{+\infty} \frac{\widetilde{G_1(u, w)}(n, \tau)}{\langle q_+(n, \tau) \rangle} d\tau \right)^2 \right)^{\frac{1}{2}} \leq cf(u, w), \tag{3.22}$$

and

$$\|G_1(u, w)\|_{(1, s, -\frac{1}{2})} \leq cf(u, w), \tag{3.23}$$

where

$$\begin{aligned} f(u, w) = & (\|u\|_{(1,s,\frac{1}{2}-\theta)}^2 + \|w\|_{(1,s,\frac{1}{2}-\theta)}^2) \|u\|_{(1,s,\frac{1}{2})} \\ & + \|u\|_{(1,s,\frac{1}{2}-\theta)} \|w\|_{(1,s,\frac{1}{2}-\theta)} \|u\|_{(1,s,\frac{1}{2})} \\ & + \left(\|u\|_{(2,\frac{1}{2},0)}^2 + \|w\|_{(2,\frac{1}{2},0)}^2 \right) \|u\|_{(1,s,0)} + \|u\|_{(2,\frac{1}{2},0)} \|w\|_{(2,\frac{1}{2},0)} \|u\|_{(1,s,0)}. \end{aligned} \quad (3.24)$$

Proof. The parameter $\frac{a}{3}$ is not an integer because we need that the third factor in the right hand side of (3.17) never vanishes. Then we have $|n - n_2 - q/3| \sim \langle n - n_2 \rangle$. Note also that $|n - n_1| \sim \langle n - n_1 \rangle$ and $|n_1 + n_2| \sim \langle n_1 + n_2 \rangle$ for $n - n_1 \neq 0$ and $n_1 + n_2 \neq 0$, respectively.

From the Cauchy-Schwarz inequality, the left hand side of (3.22) is bounded by

$$\left(\sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \int_{-\infty}^{+\infty} \frac{|\widetilde{G_1(u, w)}(n, \tau)|^2}{\langle q_+(n, \tau) \rangle^{2(1-a)}} d\tau \int_{-\infty}^{+\infty} \frac{d\tau}{\langle q_+(n, \tau) \rangle^{2a}} \right)^{\frac{1}{2}}, \quad (3.25)$$

where a will be determined later. Consider first $\widetilde{G_1(u, w)}(n, \tau) = R_2(n, \tau)$. In this case, (3.25) is bounded by

$$\begin{aligned} & \left(\sum_n \sum_{(n_1, n_2) \in H_n^1} \left(\int_{\mathbb{R}^3} \frac{\langle n \rangle^{2s} |n_1|^2}{\langle q_+(n, \tau) \rangle^{2(1-a)}} |\widetilde{u}(n_1, \tau_1)|^2 \right. \right. \\ & \quad \times \left. \left. |\widetilde{w}(n_2, \tau_2)|^2 |\widetilde{w}(n_3, \tau_3)|^2 d\tau_1 d\tau_2 d\tau \right) I_a \right)^{\frac{1}{2}} \\ = & \left(\sum_n \sum_{(n_1, n_2) \in H_n^1} \int_{\mathbb{R}^3} \frac{\langle n \rangle^{2s} |n_1|^2}{\langle q_+(n, \tau) \rangle^{2(1-a)}} \frac{g(n_1, \tau_1) h(n_2, \tau_2) p(n_3, \tau_3)}{\langle n_1 \rangle^{2s} \langle n_2 \rangle^{2s} \langle n_3 \rangle^{2s}} \right. \\ & \quad \times \left. \frac{d\tau_1 d\tau_2 d\tau}{\langle q_+(n_1, \tau_1) \rangle^{1-2\theta} \langle q_-(n_2, \tau_2) \rangle^{1-2\theta} \langle q_+(n_3, \tau_3) \rangle^{1-2\theta}} I_a \right)^{\frac{1}{2}}, \end{aligned} \quad (3.26)$$

where

$$\begin{aligned} I_a &= \int_{-\infty}^{+\infty} \frac{d\tau}{\langle q_+(n, \tau) \rangle^{2a}}, \\ g(n_1, \tau_1) &= \langle n_1 \rangle^{2s} \langle q_+(n_1, \tau_1) \rangle^{1-2\theta} |\widetilde{u}(n_1, \tau_1)|^2, \\ h(n_2, \tau_2) &= \langle n_2 \rangle^{2s} \langle q_-(n_2, \tau_2) \rangle^{1-2\theta} |\widetilde{w}(n_2, \tau_2)|^2 \end{aligned}$$

and

$$p(n_3, \tau_3) = \langle n_3 \rangle^{2s} \langle q_+(n_3, \tau_3) \rangle^{1-2\theta} |\widetilde{w}(n_3, \tau_3)|^2.$$

To estimate (3.26) we divide the region of integration in four parts:

$$\begin{aligned} A_1 &= \{(\tau_1, \tau_2, \tau) : |q_+(n, \tau)| \geq \max\{|q_+(n_1, \tau_1)|, |q_-(n_2, \tau_2)|, |q_+(n_3, \tau_3)|\}\}, \\ A_2 &= \{(\tau_1, \tau_2, \tau) : |q_+(n_1, \tau_1)| \geq \max\{|q_+(n, \tau)|, |q_-(n_2, \tau_2)|, |q_+(n_3, \tau_3)|\}\}, \\ A_3 &= \{(\tau_1, \tau_2, \tau) : |q_-(n_2, \tau_2)| \geq \max\{|q_+(n_1, \tau_1)|, |q_+(n, \tau)|, |q_+(n_3, \tau_3)|\}\}, \\ A_4 &= \{(\tau_1, \tau_2, \tau) : |q_+(n_3, \tau_3)| \geq \max\{|q_+(n_1, \tau_1)|, |q_-(n_2, \tau_2)|, |q_+(n, \tau)|\}\}. \end{aligned}$$

We also consider the sum in n_1 and n_2 of (3.26) in the following three cases

$$M_1 \geq L \geq \frac{|n|}{5} > M_2, \quad (3.27)$$

$$M_1 \geq \frac{2|n|}{3} \geq \frac{|n|}{5} > L \geq M_2, \quad (3.28)$$

$$M_1 \geq L \geq M_2 \geq \frac{|n|}{5} \quad (3.29)$$

and I_a is bounded as

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{\langle n^2 + q_+(n, \tau) \rangle^{2a} d\tau}{\langle n^2 + q_+(n, \tau) \rangle^{2a} \langle q_+(n, \tau) \rangle^{2a}} &\leq \int_{-\infty}^{+\infty} \frac{c \langle q_+(n, \tau) \rangle^{2a} d\tau}{\langle n^2 + q_+(n, \tau) \rangle^{2a} \langle q_+(n, \tau) \rangle^{2a}} \\ &\leq c \int_{-\infty}^{+\infty} \frac{d\tau}{\langle n^2 + q_+(n, \tau) \rangle^{2a}} \\ &\leq \frac{c}{\langle n \rangle^{4a-2}}, \quad \text{for } a > \frac{1}{2}. \end{aligned} \quad (3.30)$$

The first inequality in (3.30) follows from $|q_+(n, \tau)| \geq cn^2 \langle M_2 \rangle \geq cn^2$ and the last inequality is a consequence of

$$\int_{\mathbb{R}} \frac{d\tau}{\langle \tau \rangle^\alpha \langle \tau - \theta \rangle^\beta} \leq \frac{c}{\langle \theta \rangle^d}, \quad \text{with,} \quad d = \min\{\alpha, \beta, \alpha + \beta - 1\}. \quad (3.31)$$

Then, in the case (3.27) and in the region A_1 , we bound (3.26) by

$$\begin{aligned} \sup_{n, \tau} \left[\frac{\langle n \rangle^{s-2a+1}}{\langle q_+(n, \tau) \rangle^{(1-a)}} \left(\sum_{(n_1, n_2) \in A_{n, \tau}} \frac{|n_1|^2}{\langle n_1 \rangle^{2s} \langle n_2 \rangle^{2s} \langle n_3 \rangle^{2s}} H_\theta(n, n_1, n_2) \right)^{1/2} \right] \\ \times c \|u\|_{(1, s, \frac{1}{2}-\theta)} \|w\|_{(1, s, \frac{1}{2}-\theta)}^2, \end{aligned} \quad (3.32)$$

where

$$\begin{aligned} A_{n, \tau} = \left\{ (n_1, n_2) : M_1 \geq L \geq \frac{|n|}{5} > M_2, n \neq n_1, n_1 \neq -n_2, \right. \\ \left. |q_+(n, \tau)| \geq |n_1 + n_2| |n - n_1| |n - n_2 - \frac{q}{3}| \right\}, \quad \text{and} \end{aligned}$$

$$H_\theta(n, n_1, n_2) = \int_{\mathbb{R}^2} \frac{d\tau_1 d\tau_2}{\langle q_+(n_1, \tau_1) \rangle^{1-2\theta} \langle q_-(n_2, \tau_2) \rangle^{1-2\theta} \langle q_+(n_3, \tau_3) \rangle^{1-2\theta}}$$

Using the identity (3.17) and inequality (3.31), we bounded $H_\theta(n, n_1, n_2)$ by

$$\begin{aligned} &\int_{\mathbb{R}} \frac{d\tau_1}{\langle q_+(n_1, \tau_1) \rangle^{1-2\theta} \langle q_+(n_1, \tau_1) - [q_+(n, \tau) - 3(n_1 + n_2)(n - n_1)(n - n_2 - \frac{q}{3})] \rangle^{1-4\theta}} \\ &\leq \frac{c}{\langle q_+(n, \tau) - 3(n_1 + n_2)(n - n_1)(n - n_2 - \frac{q}{3}) \rangle^{1-6\theta}} \\ &\leq \frac{c}{\langle q_+(n, \tau) - 3(n_1 + n_2)(n - n_1)(n - n_2 - \frac{q}{3}) \rangle^{1-\varepsilon}}, \end{aligned} \quad (3.33)$$

for $\theta \in (0, \frac{1}{12})$, and $\varepsilon \in (6\theta, \frac{1}{2})$.

Then, (3.32) is estimate by

$$\begin{aligned}
& c \sup_{n, \tau} \left[\frac{\langle n \rangle^{s-2a+1}}{\langle q_+(n, \tau) \rangle^{(1-a)}} \left(\sum_{(n_1, n_2) \in A_{n, \tau}} \frac{|n_1|^2}{\langle n_1 \rangle^{2s} \langle n_2 \rangle^{2s} \langle n_3 \rangle^{2s}} \right. \right. \\
& \quad \times \left. \frac{1}{\langle q_+(n, \tau) - 3(n_1 + n_2)(n - n_1)(n - n_2 - \frac{q}{3}) \rangle^{1-\varepsilon}} \right)^{\frac{1}{2}} \Big] \\
& \times \|u\|_{(1, s, \frac{1}{2}-\theta)} \|w\|_{(1, s, \frac{1}{2}-\theta)}^2 \\
& = c \sup_{n, \tau} (I_1)^{\frac{1}{2}} \|u\|_{(1, s, \frac{1}{2}-\theta)} \|w\|_{(1, s, \frac{1}{2}-\theta)}^2 \leq c \|u\|_{(1, s, \frac{1}{2})} \|w\|_{(1, s, \frac{1}{2}-\theta)}^2,
\end{aligned} \tag{3.34}$$

where

$$\begin{aligned}
I_1 &= \frac{\langle n \rangle^{2s-4a+2}}{\langle q_+(n, \tau) \rangle^{2(1-a)}} \left(\sum_{(n_1, n_2) \in A_{n, \tau}} \frac{|n_1|^2}{\langle n_1 \rangle^{2s} \langle n_2 \rangle^{2s} \langle n_3 \rangle^{2s}} \right. \\
& \quad \times \left. \frac{1}{\langle q_+(n, \tau) - 3(n_1 + n_2)(n - n_1)(n - n_2 - \frac{q}{3}) \rangle^{1-\varepsilon}} \right).
\end{aligned}$$

The proof that $I_1 \leq c$ can be found in [17], Lemma 4.3. The cases (3.28) and (3.29) are analogous.

Now, we bound (3.25) in the region A_2 with $a > \frac{1}{2}$ and $\widetilde{G_1(u, w)}(n, \tau) = R_2(n, \tau)$,

$$c \left(\sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \int_{\mathbb{R}} \frac{|R_2(n, \tau)|^2}{\langle q_+(n, \tau) \rangle^{2(1-a)}} d\tau \right)^{\frac{1}{2}} = c \left\| \frac{\langle n \rangle^s R_2(n, \tau)}{\langle q_+(n, \tau) \rangle^{(1-a)}} \right\|_{l_n^2 L_\tau^2}. \tag{3.35}$$

By duality, (3.35) is equal to

$$c \sup_{\|h(n, \tau)\|_{l_n^2 L_\tau^2}} \left| \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \frac{\langle n \rangle^s R_2(n, \tau) \overline{h(n, \tau)}}{\langle q_+(n, \tau) \rangle^{(1-a)}} d\tau \right| = c \sup_{\|h(n, \tau)\|_{l_n^2 L_\tau^2}} |A(h)|. \tag{3.36}$$

Note that

$$\begin{aligned}
|A(h)| &= \left| \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \frac{\langle n \rangle^s R_2(n, \tau) \overline{h(n, \tau)}}{\langle q_+(n, \tau) \rangle^{(1-a)}} d\tau \right| \\
&\leq c \sum_{n \in \mathbb{Z}} \langle n \rangle^s \left(\int_{\mathbb{R}} \sum_{(n_1, n_2) \in H_n^1} \frac{|n_1| |\overline{h(n, \tau)}|}{\langle q_+(n, \tau) \rangle^{(1-a)}} \iint_{\mathbb{R}^2} \frac{g(n_1, \tau_1) p(n_2, \tau_2)}{\langle n_1 \rangle^s \langle n_2 \rangle^s \langle n_3 \rangle^s} \right. \\
& \quad \times \left. \frac{r(n_3, \tau_3) d\tau_1 d\tau_2 d\tau}{\langle q_+(n_1, \tau_1) \rangle^{\frac{1}{2}} \langle q_-(n_2, \tau_2) \rangle^{\frac{1}{2}-\theta} \langle q_+(n_3, \tau_3) \rangle^{\frac{1}{2}-\theta}} \right) \\
&\leq c \sup_{n_1, \tau_1} \left(\sum_{n \in \mathbb{Z}} \langle n \rangle^s \int_{\mathbb{R}} \sum_{n_2} \frac{|n_1| |\overline{h(n, \tau)}|}{\langle n_1 \rangle^s \langle q_+(n_1, \tau_1) \rangle^{\frac{1}{2}}} \int_{\mathbb{R}} \frac{p(n_2, \tau_2)}{\langle n_2 \rangle^s \langle n_3 \rangle^s} \right. \\
& \quad \times \left. \frac{d\tau_2 d\tau}{\langle q_+(n, \tau) \rangle^{(1-a)} \langle q_-(n_2, \tau_2) \rangle^{\frac{1}{2}-\theta} \langle q_+(n_3, \tau_3) \rangle^{\frac{1}{2}-\theta}} \right) \\
&\times \|g\|_{l_{n_1}^2 L_{\tau_1}^2} \|r\|_{l_{n_1}^2 L_{\tau_1}^2},
\end{aligned} \tag{3.37}$$

where

$$\begin{aligned} g(n_1, \tau_1) &= \langle n_1 \rangle^{2s} \langle q_+(n_1, \tau_1) \rangle^{\frac{1}{2}} |\tilde{u}(n_1, \tau_1)|, \\ p(n_2, \tau_2) &= \langle n_2 \rangle^s \langle q_-(n_2, \tau_2) \rangle^{\frac{1}{2}-\theta} |\tilde{w}(n_2, \tau_2)| \end{aligned}$$

and

$$r(n_3, \tau_3) = \langle n_3 \rangle^s \langle q_+(n_3, \tau_3) \rangle^{\frac{1}{2}-\theta} |\tilde{w}(n_3, \tau_3)|.$$

Therefore we can bound (3.39) by

$$\begin{aligned} & c \sup_{n_1, \tau_1} \left(\frac{|n_1|}{\langle n_1 \rangle^s \langle q_+(n_1, \tau_1) \rangle^{\frac{1}{2}}} \left(\sum_{(n, n_2) \in D_{n_1, \tau_1}} \int_{\mathbb{R}^2} \frac{\langle n \rangle^{2s}}{\langle n_2 \rangle^{2s} \langle n_3 \rangle^{2s}} \right. \right. \\ & \quad \times \frac{d\tau_2 d\tau}{\langle q_+(n, \tau) \rangle^{2(1-a)} \langle q_-(n_2, \tau_2) \rangle^{1-2\theta} \langle q_+(n_3, \tau - \tau_3) \rangle^{1-2\theta}} \Big)^{\frac{1}{2}} \Big) \\ & \quad \times \|g\|_{l_{n_1}^2 L_{\tau_1}^2} \|r\|_{l_{n_1}^2 L_{\tau_1}^2} \|p\|_{l_{n_2}^2 L_{\tau_2}^2} \|\bar{h}\|_{l_n^2 L_{\tau}^2}, \end{aligned}$$

where

$$\begin{aligned} D_{n_1, \tau_1} &= \{(n, n_2) : n \neq n_1, n_1 \neq -n_2, \\ & \quad |q_+(n_1, \tau_1)| \geq |n_1 + n_2| |n - n_1| |n - n_2 - \frac{q}{3}|\}. \end{aligned}$$

It follows from (3.35)-(3.38) that (3.25) is bounded by

$$\begin{aligned} & c \sup_{n_1, \tau_1} \left(\frac{|n_1|}{\langle n_1 \rangle^s \langle q_+(n_1, \tau_1) \rangle^{\frac{1}{2}}} \left(\sum_{(n, n_2) \in D_{n_1, \tau_1}} \int_{\mathbb{R}^2} \frac{\langle n \rangle^{2s}}{\langle n_2 \rangle^{2s} \langle n_3 \rangle^{2s}} \right. \right. \\ & \quad \times \frac{d\tau_2 d\tau}{\langle q_+(n, \tau) \rangle^{2(1-a)} \langle q_-(n_2, \tau_2) \rangle^{1-2\theta} \langle q_+(n_3, \tau_3) \rangle^{1-2\theta}} \Big)^{\frac{1}{2}} \Big) \\ & \quad \times c \|u\|_{(1, s, \frac{1}{2})} \|w\|_{(1, s, \frac{1}{2}-\theta)}^2 \\ & = c \sup_{n_1, \tau_1} (I_2)^{\frac{1}{2}} \|u\|_{(1, s, \frac{1}{2})} \|w\|_{(1, s, \frac{1}{2}-\theta)}^2 \leq c \|u\|_{(1, s, \frac{1}{2})} \|w\|_{(1, s, \frac{1}{2}-\theta)}^2, \end{aligned}$$

where

$$\begin{aligned} I_2 &= \frac{\|n_1\|^2}{\langle n_1 \rangle^{2s} \langle q_+(n_1, \tau_1) \rangle^1} \left(\sum_{(n, n_2) \in D_{n_1, \tau_1}} \int_{\mathbb{R}^2} \frac{\langle n \rangle^{2s}}{\langle n_2 \rangle^{2s} \langle n_3 \rangle^{2s}} \right. \\ & \quad \times \frac{d\tau_2 d\tau}{\langle q_+(n, \tau) \rangle^{2(1-a)} \langle q_-(n_2, \tau_2) \rangle^{1-2\theta} \langle q_+(n_3, \tau_3) \rangle^{1-2\theta}} \Big). \end{aligned}$$

The proof that $I_2 \leq c$ is similar to Lemma 4.3 in [17]. The case $\widetilde{G_1(u, w)}(n, \tau) = R_1(n, \tau)$ is similar to (3.35). In the case $\widetilde{G_1(u, w)}(n, \tau) = R_4(n, \tau)$, (3.25) is bounded by

$$\begin{aligned} & c \left\{ \sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \int_{\mathbb{R}} \left(\sum_{n_1 \in \mathbb{Z}} \iint_{\mathbb{R}^2} |n_1| |\tilde{u}(n_1, \tau_1)| |\tilde{w}(-n_1, \tau_2)| |\tilde{w}(n, \tau_3)| d\tau_1 d\tau_2 \right)^2 d\tau \right\}^{1/2} \\ & \leq c \left\{ \sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \left(\sum_{n_1 \in \mathbb{Z}} \int_{\mathbb{R}} \left(\iint_{\mathbb{R}^2} |n_1| |\tilde{u}(n_1, \tau_1)| |\tilde{w}(-n_1, \tau_2)| |\tilde{w}(n, \tau_3)| d\tau_1 d\tau_2 \right)^2 d\tau \right) \right\}^{1/2}. \end{aligned} \tag{3.38}$$

Using Minkowski's inequality, (3.38) is bounded by

$$\begin{aligned}
& \left(\sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \int_{\mathbb{R}} |\tilde{w}(n, \tau_3)|^2 d\tau \right)^{\frac{1}{2}} \left(\sum_{n_1 \in \mathbb{Z}} \iint_{\mathbb{R}^2} |n_1| |\tilde{u}(n_1, \tau_1)| |\tilde{w}(-n_1, \tau_2)| d\tau_1 d\tau_2 \right) \\
& \leq c \left(\sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \int_{\mathbb{R}} |\tilde{w}(n, \tau_3)|^2 d\tau \right)^{\frac{1}{2}} \left(\sum_{n_1 \in \mathbb{Z}} \left[\int_{\mathbb{R}} \langle n_1 \rangle^{\frac{1}{2}} |\tilde{u}(n_1, \tau_1)| d\tau_1 \right] \right. \\
& \quad \times \left. \left[\int_{\mathbb{R}} \langle n_1 \rangle^{\frac{1}{2}} |\tilde{w}(-n_1, \tau_2)| d\tau_2 \right] \right) \\
& \leq c \|\langle n \rangle^s \tilde{w}(n, \tau)\|_{l_n^2 L_{\tau}^2} \|\langle n_1 \rangle^{\frac{1}{2}} \tilde{u}(n_1, \tau)\|_{l_{n_1}^2 L_{\tau}^1} \|\langle n_1 \rangle^{\frac{1}{2}} \tilde{w}(n_1, \tau)\|_{l_{n_1}^2 L_{\tau}^1} \\
& = c \|w\|_{(1,s,0)} \|u\|_{(2,\frac{1}{2})} \|w\|_{(2,\frac{1}{2})}.
\end{aligned}$$

The case $\widetilde{G_1(u, w)}(n, \tau) = R_3(n, \tau)$ is similar to $R_4(n, \tau)$, for $w = u$. In the case of $\widetilde{G_1(u, w)}(n, \tau) = R_6(n, \tau)$ we bound (3.25) by

$$\begin{aligned}
& c \left(\sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \int_{\mathbb{R}} \left(\iint_{\mathbb{R}^2} |n| |\tilde{u}(n, \tau_1)| |\tilde{w}(-n, \tau_2)| |\tilde{w}(n, \tau - \tau_1 - \tau_2)| d\tau_1 d\tau_2 \right)^2 d\tau \right)^{\frac{1}{2}} \\
& \leq c \left(\sum_{n \in \mathbb{Z}} \left[\iint_{\mathbb{R}^2} |n| |\tilde{u}(n, \tau_1)| |\tilde{w}(-n, \tau_2)| d\tau_1 d\tau_2 \right]^2 \right)^{\frac{1}{2}} \left(\sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \int_{\mathbb{R}} |\tilde{w}(n, \tau_3)|^2 d\tau \right)^{\frac{1}{2}} \\
& \leq c \|\langle n \rangle^s \tilde{w}(n, \tau)\|_{l_n^2 L_{\tau}^2} \|\langle n \rangle^{\frac{1}{2}} \tilde{u}(n, \tau_1)\|_{l_n^2 L_{\tau_1}^1} \|\langle n \rangle^{\frac{1}{2}} \tilde{w}(n, \tau_2)\|_{l_n^2 L_{\tau_2}^1} \\
& = c \|w\|_{(1,s,0)} \|u\|_{(2,\frac{1}{2},0)} \|w\|_{(2,\frac{1}{2},0)}.
\end{aligned}$$

For the cases $\widetilde{G_1(u, w)}(n, \tau) = R_i(n, \tau)$, $i = 7, \dots, 12$, we follow a similar argument.

The proof of (3.23) follows the same lines as (3.22), choosing $a = \frac{1}{2}$ and not considering I_a , because

$$\|G_1(u, w)\|_{(1,s,-\frac{1}{2})} = \left(\sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \int_{-\infty}^{+\infty} \frac{|\widetilde{G_1(u, w)}(n, \tau)|^2}{\langle q_+(n, \tau) \rangle} d\tau \right)^{\frac{1}{2}}. \quad (3.39)$$

□

The following lemma is found in [17], Lemma 4.6.

Lemma 3.3. *For $s \in \mathbb{R}$, $0 < \varepsilon < \frac{1}{2}$, $T \in (0, 1)$ and $0 < \theta' < \theta < \frac{1}{2}$, we have the following inequalities*

$$\|\psi_T(t)f\|_{(1,s,\frac{1}{2})} \leq c(\varepsilon)T^{-\varepsilon}\|f\|_{(1,s,\frac{1}{2})},$$

$$\|\psi_T(t)f\|_{(1,s,\frac{1}{2}-\theta)} \leq cT^{\theta-\theta'}\|f\|_{(1,s,\frac{1}{2})},$$

$$\|\psi_T(t)f\|_{(2,s,0)} \leq c\|f\|_{(2,s,0)}.$$

Now we are able to prove the main theorem of this section.

Proof of Theorem 1.6. Let $s \geq \frac{1}{2}$. With all the estimates at hand, for $T \in (0, 1)$, define

$$Y_T^a = \{ \vec{v} \in Y_s \times Y_s : \|\vec{v}\|_{Y_s \times Y_s} \leq a \},$$

$$\begin{aligned}
\Phi(\vec{u}) &= \Psi(t)W_p(t)\vec{u}_0 - \Psi(t) \int_0^t W_p(t-t')G(\Psi_T(t')\vec{u}(t'))dt' \\
&= \begin{pmatrix} \psi(t)S_p(t)u_0 - \psi(t) \int_0^t S_p(t-t')G_1(\psi_T u, \psi_T w)(t')dt' \\ \psi(t)S_p(t)w_0 - \psi(t) \int_0^t S_p(t-t')G_1(\psi_T w, \psi_T u)(t')dt' \end{pmatrix}.
\end{aligned} \tag{3.40}$$

From the lemmata 4.2 and 4.5 we have that

$$\begin{aligned}
\|\Phi(\vec{u})\|_{Y_s \times Y_s} &\leq c\|\vec{u}_0\|_{H^s \times H^s} + f(\psi_T u, \psi_T w) \\
&\leq c\|\vec{u}_0\|_{H^s \times H^s} + c \left(\|\psi_T u\|_{(1,s,\frac{1}{2}-\theta)}^2 + \|\psi_T w\|_{(1,s,\frac{1}{2}-\theta)}^2 \right) \|\psi_T u\|_{(1,s,\frac{1}{2})} \\
&\quad + c\|\psi_T u\|_{(1,s,\frac{1}{2}-\theta)} \|\psi_T w\|_{(1,s,\frac{1}{2}-\theta)} \|\psi_T u\|_{(1,s,\frac{1}{2})} \\
&\quad + c \left(\|\psi_T u\|_{(2,\frac{1}{2},0)}^2 + \|\psi_T w\|_{(2,\frac{1}{2},0)}^2 \right) \|\psi_T u\|_{(1,s,0)} \\
&\quad + c\|\psi_T u\|_{(2,\frac{1}{2},0)} \|\psi_T w\|_{(2,\frac{1}{2},0)} \|\psi_T u\|_{(1,s,0)} \\
&\leq c\|\vec{u}_0\|_{H^s \times H^s} + cT^{\theta-\theta'+\varepsilon} \|\vec{u}\|_{Y_s \times Y_s}^3 \leq c\|\vec{u}_0\|_{H^s \times H^s} + cT^{\theta-\theta'+\varepsilon} a^3,
\end{aligned}$$

for $\vec{u} \in Y_T^a$ and $0 < \theta' - \varepsilon < \theta < \frac{1}{12}$. Analogously, for $\vec{u}, \vec{v} \in Y_T^a$ we have that

$$\|\Phi(\vec{u}) - \Phi(\vec{v})\|_{Y_s \times Y_s} \leq cT^{\theta-\theta'+\varepsilon} a^2 \|\vec{u} - \vec{v}\|_{Y_s \times Y_s}. \tag{3.41}$$

We choose $\frac{a}{2} = c\|\vec{u}_0\|_{H^s}$ and T small enough, such that $cT^{\theta-\theta'+\varepsilon} a^3 \leq \frac{a}{2}$, then $\|\Phi(\vec{u})\|_{Y_s \times Y_s} \leq a$ and Φ is a contraction. Uniqueness and continuous dependence follow in standard way.

4. PROOF OF THEOREM 1.13

First we obtain conserved quantities for (1.1) with $\sigma_\alpha = \sigma_\beta = \sigma_\mu = 1$.

For $\Omega = \mathbb{T}$, define

$$H_0(u, w) = 2\mu \operatorname{Im} \left(\int_{\Omega} ((u\bar{u}_x)^2 + (w\bar{w}_x)^2) dx \right) + 4\mu \operatorname{Im} \left(\int_{\Omega} (u\bar{u}_x w\bar{w}_x) dx \right),$$

$$\begin{aligned}
H_1(u, w) &= (\beta + 2\mu) \int_{\Omega} [|u_x|^2 \partial_x(|u|^2) + |w_x|^2 \partial_x(|w|^2)] dx \\
&\quad + (\beta + 2\mu) \int_{\Omega} [|u_x|^2 \partial_x(|w|^2)] dx \\
&\quad + (\beta + 2\mu) \int_{\Omega} [|w_x|^2 \partial_x(|u|^2)] dx + 4\alpha \operatorname{Im} \left(\int_{\Omega} (u\bar{u}_x w\bar{w}_x) dx \right) \\
&\quad + 2\alpha \operatorname{Im} \left(\int_{\Omega} ((u\bar{u}_x)^2 + (w\bar{w}_x)^2) dx \right),
\end{aligned}$$

$$\begin{aligned}
H_2(u, v) &= \left(-\frac{1}{2}\beta + 2\mu \right) \int_{\Omega} (\partial_x(|u|^2) |w|^4 + \partial_x(|w|^2) |u|^4) dx \\
&\quad + q \operatorname{Im} \int_{\Omega} ((u\bar{u}_x)^2 + (w\bar{w}_x)^2) dx \\
&\quad + \frac{3}{2}\gamma \int_{\Omega} [|u_x|^2 \partial_x(|u|^2) + |w_x|^2 \partial_x(|w|^2)] dx,
\end{aligned}$$

$$\begin{aligned}
H_3(u, w) &= \left(\frac{1}{2}\beta - 2\mu \right) \int_{\Omega} (\partial_x (|u|^2) |w|^4 + \partial_x (|w|^2) |u|^4) dx \\
&\quad + 2q \operatorname{Im} \left(\int_{\Omega} (u \bar{u}_x w \bar{w}_x) dx \right) \\
&\quad + \frac{3}{2} \gamma \int_{\Omega} [|u_x|^2 \partial_x (|w|^2) + |w_x|^2 \partial_x (|u|^2)] dx.
\end{aligned}$$

Lemma 4.1. *Let $\vec{u}_0 = (u_0, w_0) \in H^{s'}(\Omega) \times H^{s'}(\Omega)$ with s' large enough and $\vec{u} \in C([-T, T]; H^{s'}(\Omega) \times H^{s'}(\Omega))$ solutions of (1.1) with $\sigma_{\alpha} = \sigma_{\beta} = \sigma_{\mu} = 1$. Then*

$$i \partial_t \left(\int_{\Omega} (u \bar{u}_x + w \bar{w}_x) dx \right) = H_0(u, w), \quad (4.1)$$

$$\partial_t \left(\|u_x\|_{L^2(\Omega)}^2 + \|w_x\|_{L^2(\Omega)}^2 \right) = -H_1(u, w), \quad (4.2)$$

$$\frac{1}{2} \partial_t \left(\|u_x\|_{L^4(\Omega)}^4 + \|w_x\|_{L^4(\Omega)}^4 \right) = -H_2(u, w), \quad (4.3)$$

$$\partial_t \int_{\Omega} |u|^2 |w|^2 dx = -H_3(u, w). \quad (4.4)$$

Proof. Multiply the first equation in (1.1) by \bar{u}_x , second equation by \bar{w}_x , integrate in x and take the real part to obtain

$$\operatorname{Re} \left(2i \int_{\Omega} u_t \bar{u}_x dx \right) = i \int_{\Omega} (u_t \bar{u}_x - \bar{u}_t u_x) dx = i \partial_t \left(\int_{\Omega} u \bar{u}_x dx \right),$$

$$\begin{aligned}
\operatorname{Re} \left(2i\mu \int_{\Omega} u \partial_x (|u|^2) \bar{u}_x dx \right) &= -\operatorname{Im} \left(2\mu \int_{\Omega} u \partial_x (|u|^2) \bar{u}_x dx \right) \\
&= -2\mu \operatorname{Im} \left(\int_{\Omega} (u \bar{u}_x)^2 dx \right),
\end{aligned}$$

$$\begin{aligned}
&\operatorname{Re} \left(2i\mu \int_{\Omega} u \partial_x (|w|^2) \bar{u}_x dx \right) + \operatorname{Re} \left(2i\mu \int_{\Omega} w \partial_x (|u|^2) \bar{w}_x dx \right) \\
&= -4\mu \operatorname{Im} \left(\int_{\Omega} (u \bar{u}_x w \bar{w}_x) dx \right),
\end{aligned}$$

$$\begin{aligned}
&\operatorname{Re} \left(2\alpha \int_{\Omega} u (|u|^2 + |w|^2) \bar{u}_x dx \right) + \operatorname{Re} \left(2\alpha \int_{\Omega} w (|u|^2 + |w|^2) \bar{w}_x dx \right) \\
&= -\alpha \int_{\Omega} \partial_x (|w|^2) |u|^2 dx + \alpha \int_{\Omega} \partial_x (|u|^2) |w|^2 dx = 0.
\end{aligned}$$

The equality (4.1) is obtained by adding the previous resulting equations.

To obtain (4.2) multiply the first equation in (1.1) by \bar{u}_{xx} , the second equation by \bar{w}_{xx} , integrate in x , take the imaginary part and add the resulting equations. The equality (4.3) is obtained multiplying the first equation in (1.1) by $|u|^2 \bar{u}$, the second equation by $|w|^2 \bar{w}$, integrate in x , take the imaginary part and add the resulting equations. Similarly, we obtain (4.4) multiplying the first equation of (1.1) by $|w|^2 \bar{u}$, the second equation by $|u|^2 \bar{w}$, integrate in x , take the imaginary part and sum the resulting equations. \square

Lemma 4.2. *Let $\vec{u}_0 \in H^1(\Omega) \times H^1(\Omega)$ and $\vec{u} \in C([-T, T]; H^1(\Omega) \times H^1(\Omega))$ solution of (1.1). Then,*

$$I_1(u, w) = \|u\|_{L^2(\Omega)}^2 + \|w\|_{L^2(\Omega)}^2 = I_1(u_0, w_0), \quad (4.5)$$

$$\begin{aligned} I_2(u, v) &= i(-3\gamma\alpha + \beta q + 2\mu q) \int_{\Omega} (u\bar{u}_x + w\bar{w}_x) dx + \frac{3}{2}\gamma \int_{\Omega} (|u_x|^2 + |w_x|^2) dx \\ &\quad + \frac{1}{2}(\beta + 2\mu) \int_{\Omega} (|u|^4 + |w|^4) dx + (\beta + 2\mu) \int_{\Omega} |u|^2 |w|^2 dx \\ &= I_2(u_0, w_0). \end{aligned} \quad (4.6)$$

Proof. The following combination of (4.1)-(4.4) lead to (4.6)

$$\left(\frac{-3\gamma\alpha + (\beta + 2\mu)q}{2\mu} \right) (4.1) + \frac{3\gamma}{2} (4.2) + (\beta + 2\mu) (4.3) + (\beta + 2\mu) (4.4).$$

□

The global solution in Theorem 1.13 follows from Theorem 1.6 and (4.5)-(4.6).

Final Remark

An interesting mathematical problem would be to study the local well-posedness on the torus of the related higher-order nonlinear Schrödinger system

$$\begin{cases} 2i\partial_t u + q_1 \partial_x^2 u + i\gamma_1 \partial_x^3 u = F_1(u, w) \\ 2i\partial_t w + q_2 \partial_x^2 w + i\gamma_2 \partial_x^3 w = F_2(u, w), \\ u(x, 0) = u_0, w(x, 0) = w_0, \end{cases}$$

when $q_1 \neq q_2$ and $\gamma_1 \neq \gamma_2$.

Note that in this case the resonant regions would be different from the ones in [3], so the proofs of [3] would need to be carefully modified.

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